

Initial value problems in Clifford-type analysis *

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Abstract

We consider an initial value problem of type

$$\frac{\partial u}{\partial t} = \mathcal{F}(t, x, u, \partial_j u), \quad u(0, x) = \varphi(x),$$

where t is the time, $x \in \mathbb{R}^n$ and u_0 is a Clifford type algebra-valued function satisfying $\mathbf{D}u = \sum_{j=0}^n \lambda_j(x) e_j \partial_j u = 0$, $\lambda_j(x) \in \mathbb{R}$ for all j . We will solve this problem using the technique of associated spaces. In order to do that, we give sufficient conditions on the coefficients of the operators \mathcal{F} and \mathbf{D} , where $\mathcal{F}(u) = \sum_{i=0}^n A^{(i)}(x) \partial_i u$ for $A^{(i)}(x) \in \mathbb{R}$ or $A^{(i)}(x)$ belonging to a Clifford-type algebra, such that these operators are an associated pair.

1 Introduction

We consider the initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{F}(t, x, u, \partial_j u); \tag{1}$$

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$$u(0, x) = \varphi(x), \quad (2)$$

where t means the variable time, $x = (x_0, \dots, x_n)$ is a spacelike variable on \mathbf{R}^{n+1} and ∂_j is the operator differentiation respect to x_j . The problem (1),(2) is equivalent to the integro-differential equation (see [4])

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u(\tau, x), \partial_j u(\tau, x)) d\tau. \quad (3)$$

Therefore the solutions of the problem (1),(2) can be constructed as fixed points of the operator

$$U(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u(\tau, x), \partial_j u(\tau, x)) d\tau. \quad (4)$$

It is well known that the classical Cauchy-Kovalevskaya theorem gives a unique solution to the problem (1),(2) considered in the context of complex analysis:

$$\frac{\partial u}{\partial t} = \mathcal{F}(t, z, u, \partial_z u); \quad (5)$$

$$u(0, z) = \varphi(z), \quad (6)$$

provided $\mathcal{F}(t, z, u, \partial_z u)$ and $\varphi(z)$ are holomorphic functions in its variables. The complex valued solution $u(t, z)$ of (5), (6) is holomorphic in z and uniquely determined. H. Lewy showed that this problem has no solution if \mathcal{F} is not a holomorphic function. He constructed functions $f(t, x, y)$ infinitely many differentiable such that the equation:

$$2i(x + iy)\partial_t w = \partial_x w + i\partial_y w + f(t, x, y)$$

has no solution (see [3]). This shows that the equation (3) does not always have a solution even if $\mathcal{F}(t, x, u, \partial_j u)$ and $\varphi(x)$ are infinitely many differentiables.

The concept of associated spaces [2, 7, 9] leads to conditions under which the equation (3) has solution. This concept comes from complex analysis: In the holomorphic case (5), (6) the associated space is the space of holomorphic functions and the right hand side $\mathcal{F}(t, z, u, \partial_z u)$ transforms this space into itself.

In order to apply a fixed-point theorem, the operator (4) has to be estimated in a suitable function space whose elements depend on t and x .

This can be done by the so called interior estimate for the associated space. Such estimate describes the behaviour of the derivatives near the boundary. In case of holomorphic functions, such estimates can be obtained by the Cauchy Integral Formula. Similar estimates can be shown in the framework of Clifford analysis. This makes it possible to solve initial value problems with monogenic initial functions (see [12]). For initial value problems with other kind of initial functions see [6, 13].

Let \mathcal{F} be a differential operator while \mathcal{G} is a differential operator with respect to the space variable x whose coefficients do not depend on the time t . The operators \mathcal{F} and \mathcal{G} are said to be associated if $\mathcal{G}u = 0$ implies $\mathcal{G}(\mathcal{F}u) = 0$, for each t . The solutions of the differential equation $\mathcal{G}u = 0$ form a function space called associated space to \mathcal{G} . In [7, 8, 9] we can see that, if the initial function (2) satisfies an associated equation $\mathcal{G}u = 0$ and the elements of the associated space satisfy an interior estimate, then there exists a (uniquely determined) solution of the initial problem (1), (2) also satisfying the associated equation for each t .

In this paper, we will use this technique in order to solve the initial problem (1), (2) when the operator \mathcal{G} is given by

$$\mathbf{D}u = \sum_{j=0}^n \lambda_j(x) e_j \partial_j u. \quad (7)$$

where the functions λ_j 's are supposed to be real valued and u is a continuously differentiable function taking values in a Clifford type algebras depending on parameters. We call this operator the generalized Cauchy-Riemann operator in \mathbb{R}^{n+1} . First we will determine all \mathcal{F} in the form

$$\mathcal{F}u = \sum_{i=0}^n A^{(i)}(x) \partial_i u \quad (8)$$

for which solutions φ of $\mathbf{D}\varphi = 0$ are admissible initial functions. The functions $A^{(i)}(x)$ $i = 0, 1, \dots, n$, are functions of class \mathbf{C}^1 in a domain Ω of \mathbb{R}^{n+1} and are considered as real valued functions and as Clifford-type algebra valued. Later on, the coefficients (belonging to \mathbf{R}) of the operator \mathcal{F} are given and it is possible to obtain the conditions over the operator \mathbf{D} such that the pair $(\mathcal{F}, \mathbf{D})$ is associated.

2 Preliminaries

A Clifford algebra depending on parameters (see [11]) can be defined as equivalence classes in the ring $R[X_1, \dots, X_n]$ of polynomials in n variables X_1, \dots, X_n with real coefficients, where two polynomials are said to be equivalent if their difference is a polynomial for which each term contains at least one of the factors

$$X_j^{k_j} + \alpha_j \text{ and } X_i X_j + X_j X_i - 2\gamma_{ij}, \quad (9)$$

where $i, j = 1, \dots, n$, $i \neq j$, and the $k_j \geq 2$ are natural numbers. The parameters α_j and $\gamma_{ij} = \gamma_{ji}$ have to be real and may depend also on further variables such as the variable x in \mathbf{R}^{n+1} . If the parameters do not depend on further variables and if $n \geq 3$, the Clifford type algebra generated by the structure polynomials (9) is denoted by $\mathcal{A}_n(k_j, \alpha_j, \gamma_{ij})$. For $n = 1$ we write $\mathcal{A}_1(k, \alpha)$. In $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ all of the k_j are equal to 2 and α_j and γ_{ij} are constant. This algebra has the dimension 2^n . The classical Clifford algebra $\mathcal{A}_n(2, 1, 0)$ is denoted by \mathcal{A}_n .

In this paper the functions u will be considered in a domain Ω of \mathbf{R}^{n+1} with values in the particular algebra $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$, thus $u(x) = \sum_{A \in \Gamma} u_A(x) e_A$, where $\Gamma = \{0, 1, \dots, n, 12, \dots, 12 \dots n\}$ and each u_A is a real valued function.

We will understand that a function $u(x) = \sum_{A \in \Gamma} u_A(x) e_A$, with values in an algebra de Clifford $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ is of class \mathbf{C}^1 (or \mathbf{C}^2) in the domain Ω if each u_A is of class \mathbf{C}^1 (or \mathbf{C}^2) in Ω .

In similar form as the definition of monogenic functions in the classical algebra \mathcal{A}_n (see [10]), we have

Definition 1. *A continuously differentiable function $u(x)$ with values in the generalized Clifford algebra given by (9) satisfying $\mathbf{D}u = 0$ is called generalized (left) monogenic function.*

For the functions u of class \mathbf{C}^1 in Ω , the conjugated operator associated to generalized Cauchy-Riemman operator (7) is defined by

$$\overline{\mathbf{D}}u = \lambda_0(x) \partial_0 u - \sum_{j=1}^n \lambda_j(x) e_j \partial_j u.$$

Since $\lambda_i(x)$'s are supposed to be real valued, a monogenic function u of class \mathbf{C}^2 verifies

$$\begin{aligned}\overline{\mathbf{D}}\mathbf{D}u &= \sum_{i=0}^n \lambda_0(x) \partial_0(\lambda_i(x)) e_i \partial_i u - \sum_{j=1}^n \sum_{i=0}^n \lambda_j(x) e_j \partial_j(\lambda_i(x)) e_i \partial_i u \\ &+ \lambda_0^2(x) \partial_0^2 u + \sum_{i=0}^n \alpha_i \lambda_i^2(x) \partial_i^2 u - 2 \sum_{i < j} \gamma_{ij} \lambda_i(x) \lambda_j(x) \partial_i \partial_j u = 0\end{aligned}\quad (10)$$

If $\lambda_i(x)$ are real constants for all $i = 1, \dots, n$, then the partial differential equation (10) is elliptical provided

$$\alpha_j > 0 \quad \text{and} \quad |\gamma_{ij} \lambda_i \lambda_j| \leq k, \quad i, j = 1, \dots, n, \quad (11)$$

for a suitable constant k . Further if the underlying algebra is $\mathcal{A}_n(2, \alpha_j, 0)$ and $\lambda_i(x)$'s are real valued functions, then (10) is also an elliptic equation under the condition

$$\alpha_j > 0 \quad j = 1, \dots, n. \quad (12)$$

2.1 A formula for the product $\mathbf{D}(u \cdot v)$

Now, based on the previously seen definitions, a formula of the generalized Cauchy-Riemann operator applied to the product of two functions u and v with values in $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ is built.

Let u and v be functions defined in \mathbb{R}^{n+1} , continuously differentiable and with values in $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$. Using induction over n , we shall show the identity

$$\partial_k(u \cdot v) = \partial_k(u) \cdot v + u \cdot \partial_k(v), \quad k = 0, \dots, n. \quad (13)$$

For the case $n = 1$. Let $\Gamma_0 = \{0, 1\}$ be and $u = \sum_{i \in \Gamma_0} u_i e_i$, $v = \sum_{i \in \Gamma_0} v_i e_i$.

Then $u \cdot v = \sum_{i, j \in \Gamma_0} u_i v_j e_{ij}$ and trivially $\partial_k(u \cdot v) = \partial_k(u) \cdot v + u \cdot \partial_k(v)$.

Now we suppose that the identity (13) is valid for $n = k$. Let's consider the index sets $\Gamma = \{0, 1, 2, \dots, 12, 13, \dots, 123 \dots k + 1\}$, $\Gamma_1 = \{0, 1, 2, \dots, 12, 13, \dots, 123 \dots k\}$ and $\Gamma_2 = \Gamma - \Gamma_1$. Thus $u = u^{(1)} + u^{(2)}$ and $v = v^{(1)} + v^{(2)}$ where $u^{(i)}$, $v^{(i)}$ are a linear combination of the elements e_A of the bases indexed by the sets Γ_i , $i = 1, 2$. Then

$$\begin{aligned}\partial_k(u \cdot v) &= \partial_k[(u^{(1)} + u^{(2)})(v^{(1)} + v^{(2)})] \\ &= \partial_k(u^{(1)} v^{(1)}) + \partial_k(u^{(1)} v^{(2)}) + \partial_k(u^{(2)} v^{(1)}) + \partial_k(u^{(2)} v^{(2)}).\end{aligned}$$

Applying the induction hypothesis and rearrange the terms we obtain that $\partial_k(u \cdot v) = \partial_k(u) \cdot v + u \cdot \partial_k(v)$ and the proof is obtained. Now, according to definition (7),

$$\begin{aligned} \mathbf{D}(u \cdot v) &= \sum_{i=0}^n \lambda_i(x) e_i \partial_i(u \cdot v) = \sum_{i=0}^n \lambda_i(x) e_i (\partial_i(u) \cdot v + u \cdot \partial_i(v)) \\ &= \mathbf{D}(u) \cdot v + \sum_{i=0}^n \lambda_i(x) e_i (u \cdot \partial_i(v)). \end{aligned} \quad (14)$$

If $n = 2$, the general formula (14) yields

$$\begin{aligned} \mathbf{D}(u \cdot v) &= \mathbf{D}(u) \cdot v + u \cdot \mathbf{D}v \\ &+ 2\lambda_1[-u_2\gamma e_0 - u_{12}\gamma e_1 - u_{12}\alpha_1 e_2 + u_2 e_{12}] \partial_1 v \\ &+ 2\lambda_2[u_1\gamma e_0 + u_{12}\alpha_2 e_1 + u_{12}\gamma e_2 - u_1 e_{12}] \partial_2 v. \end{aligned} \quad (15)$$

This expression is a consequence of using

$$e_1 \cdot u = u \cdot e_1 + 2[-u_2\gamma e_0 - u_{12}\gamma e_1 - u_{12}\alpha_1 e_2 + u_2 e_{12}]$$

and

$$e_2 \cdot u = u \cdot e_2 + 2[u_1 e_0 \gamma + u_{12} \alpha_2 e_1 + u_{12} \gamma e_2 - u_1 e_{12}].$$

3 Sufficient conditions

3.1 Conditions over the coefficients of \mathcal{F}

We consider the operator $\mathcal{F}(u)$ defined by (8), where u is a $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ -valued function and we will determine conditions over $A^{(i)}$ guaranteeing

$$\mathbf{D}u = 0 \Rightarrow \mathbf{D}(\mathcal{F}u) = 0. \quad (16)$$

From the equation $\mathbf{D}u = 0$ we obtain

$$\partial_0 u(x) = - \sum_{j=1}^n \beta_j(x) e_j \partial_j u,$$

where $\beta_j(x) = \frac{\lambda_j(x)}{\lambda_0(x)}$, $j = 1, \dots, n$, $\lambda_0(x) \neq 0$. This formula leads to the equality

$$\partial_k \partial_0 u(x) = - \sum_{j=1}^n \partial_k (\beta_j(x) \cdot e_j \cdot \partial_j u(x)) \quad k = 0, 1, \dots, n.$$

From $u \in C^2$, we get $\partial_k \partial_0 u = \partial_0 \partial_k u$, $k = 1, \dots, n$. Thus both last equations make this way possible to write

$$\mathbf{D}(\partial_k u(x)) = - \sum_{j=1}^n \lambda_0 \partial_k(\beta_j(x)) \cdot e_j \cdot \partial_j u(x), \quad k = 0, 1, \dots, n. \quad (17)$$

The different conditions to be set will depend on the Cauchy-Riemann operator chosen and of the characteristics of $A^{(i)}(x)$'s.

Case I: $A^{(i)}$'s are real valued functions. Applying D to (8) and considering (17), it follows that $\mathbf{D}(\mathcal{F}u)$ can be expressed as a linear combination of the first order derivatives of u :

$$\begin{aligned} \mathbf{D}(\mathcal{F}u) &= \sum_{i=0}^n (\mathbf{D}(A^{(i)}(x)) \cdot \partial_i u + A^{(i)}(x) \cdot \mathbf{D}(\partial_i u)) \\ &= \sum_{i=1}^n (\mathbf{D}A^{(i)} - \mathbf{D}A^{(0)} \beta_i e_i) \partial_i u - \sum_{i=1}^n \sum_{j=0}^n A^{(j)}(x) \lambda_0 \partial_j(\beta_i) e_i \partial_i u \end{aligned}$$

and equating its coefficients to zero we obtain the following n sufficient conditions over the $n + 1$ real functions $A^{(i)}$:

$$\mathbf{D}A^{(i)}(x) - \mathbf{D}A^{(0)}(x) \beta_i(x) e_i - \sum_{j=0}^n A^{(j)}(x) \lambda_0(x) \partial_j \beta_i(x) e_i = 0, \quad (18)$$

for $i = 1, \dots, n$. These conditions guarantee (16).

Case II: $A^{(i)}$'s are $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ -valued functions. Suppose $A^{(i)}$ functions of class C^1 . Due to the heavy calculations to be made, we will restrict to the case $n = 2$. From (8) and $n = 2$ we have $\mathcal{F}u = \sum_{i=0}^2 A^{(i)}(x) \partial_i u$, and since $A^{(i)}$'s are defined in \mathbf{R}^3 with values in $\mathcal{A}_2(2, \alpha_1, \alpha_2, \gamma)$ formula (15) is used

$$\begin{aligned}
\mathbf{D}(\mathcal{F}u) &= \sum_{i=0}^2 \mathbf{D} [A^{(i)}(x) \cdot \partial_i u] \\
&= \sum_{i=0}^2 \mathbf{D} [A^{(i)}(x)] \cdot \partial_i u + A^{(i)}(x) \cdot \mathbf{D} [\partial_i u] \\
&\quad + 2\lambda_1 \left[-A_2^{(0)} \gamma e_0 - A_{12}^{(0)} \gamma e_1 - A_{12}^{(0)} \alpha_1 e_2 + A_2^{(0)} e_{12} \right] \partial_1 \partial_0 u \\
&\quad + 2\lambda_2 \left[A_1^{(0)} \gamma e_0 + A_{12}^{(0)} \alpha_2 e_1 + A_{12}^{(0)} \gamma e_2 - A_1^{(0)} e_{12} \right] \partial_2 \partial_0 u \\
&\quad + 2 \sum_{i=1}^2 \left\{ \lambda_1 \left[-A_2^{(i)} \gamma e_0 - A_{12}^{(i)} \gamma e_1 - A_{12}^{(i)} \alpha_1 e_2 + A_2^{(i)} e_{12} \right] \partial_1 \partial_i u \right. \\
&\quad \left. + \lambda_2 \left[A_1^{(i)} \gamma e_0 + A_{12}^{(i)} \alpha_2 e_1 + A_{12}^{(i)} \gamma e_2 - A_1^{(i)} e_{12} \right] \partial_2 \partial_i u \right\},
\end{aligned}$$

where $A^{(k)} = A_0^{(k)} + A_1^{(k)} e_1 + A_2^{(k)} e_2 + A_{12}^{(k)} e_{12}$. From (17)

$$\begin{aligned}
\mathbf{D}(\mathcal{F}u) &= \sum_{i=1}^2 (\mathbf{D} A^{(i)} - \mathbf{D} A^{(0)} \beta_i e_i) \partial_i u \\
&\quad - \sum_{i=1}^2 \sum_{j=0}^2 A^{(j)}(x) \lambda_0 \cdot \partial_j (\beta_i) \cdot e_i \cdot \partial_i u \\
&\quad - 2\lambda_1 \left[-A_2^{(0)} \gamma e_0 - A_{12}^{(0)} \gamma e_1 - A_{12}^{(0)} \alpha_1 e_2 + A_2^{(0)} e_{12} \right] \\
&\quad \left[\sum_{i=1}^2 \partial_1 (\beta_i) \cdot e_i \cdot \partial_i u + \beta_i \cdot e_i \cdot \partial_1 (\partial_i u) \right] \\
&\quad - 2\lambda_2 \left[A_1^{(0)} \gamma e_0 + A_{12}^{(0)} \alpha_2 e_1 + A_{12}^{(0)} \gamma e_2 - A_1^{(0)} e_{12} \right] \\
&\quad \left[\sum_{i=1}^2 \partial_2 (\beta_i) \cdot e_i \cdot \partial_i u + \beta_i \cdot e_i \cdot \partial_2 (\partial_i u) \right] \\
&\quad + 2 \sum_{i=1}^2 \left\{ \lambda_1 \left[-A_2^{(i)} \gamma e_0 - A_{12}^{(i)} \gamma e_1 - A_{12}^{(i)} \alpha_1 e_2 + A_2^{(i)} e_{12} \right] \partial_1 \partial_i u \right. \\
&\quad \left. + \lambda_2 \left[A_1^{(i)} \gamma e_0 + A_{12}^{(i)} \alpha_2 e_1 + A_{12}^{(i)} \gamma e_2 - A_1^{(i)} e_{12} \right] \partial_2 \partial_i u \right\}.
\end{aligned}$$

If u is assumed of class C^2 then the condition (16) is verified provided the coefficients of the first order derivatives satisfy

$$(\mathbf{D}A^{(1)} - \mathbf{D}A^{(0)}\beta_1 e_1) - \sum_{i=0}^2 A^{(i)}(x)\lambda_0 \partial_i \beta_1 \cdot e_1 = 0, \quad (19)$$

$$(\mathbf{D}A^{(2)} - \mathbf{D}A^{(0)}\beta_2 e_2) - \sum_{i=0}^2 A^{(i)}(x)\lambda_0 \partial_i \beta_2 \cdot e_2 = 0, \quad (20)$$

producing altogether 8 equations. Additional conditions for second order coefficients are

$$\begin{aligned} \alpha_1 \beta_1 A_{12}^{(0)} &= A_2^{(1)} & -\beta_1 A_2^{(0)} &= A_{12}^{(1)}, \\ \alpha_2 \beta_2 A_{12}^{(0)} &= -A_1^{(2)} & \beta_2 A_1^{(0)} &= A_{12}^{(2)}, \\ 2\beta_1 \beta_2 \gamma A_{12}^{(0)} &= \beta_2 A_1^{(1)} - \beta_1 A_2^{(2)}, \end{aligned} \quad (21)$$

and they allow reducing the number of unknown quantities in (19) and (20) to 21. Finally, conditions (19)-(21) guaranteeing that the pair of operators $\mathcal{F}u = \sum_{i=0}^2 A^{(i)}(x)\partial_i u$ and $\mathbf{D}u = \sum_{j=0}^2 \lambda_j(x)e_j \partial_j u$ be associated, have been obtained. For the classic algebra \mathcal{A}_2 and the case in which all λ_i are equal to 1 and the components $A_j^{(i)}$, $i = 1, \dots, n$, $j = 1, \dots, n$, linear functions depending of the variables x_0, x_1 and x_2 , of (19) and (20), 21-8=13 linearly independent associated operators \mathcal{F} , called admissible, are obtained.

Theorem 1. *Consider the operator \mathbf{D} defined by (7) where the coefficients $\lambda^{(i)}$ are real valued functions and of class \mathbf{C}^1 defined in $\Omega \subset \mathbb{R}^{n+1}$. Then \mathbf{D} is associated to each operator \mathcal{F} given by (8) if the conditions (18) or (19)-(21) are satisfied.*

Remark 1. *The results given in Theorem (1) are an extension of those shown in [5] for the classic algebra \mathcal{A}_2 .*

3.2 Conditions over the coefficients of \mathbf{D}

Now we consider the operator \mathcal{F} given by (8) and suppose the coefficients $A^{(i)}$ are real valued. In order to determine conditions over the real valued

functions λ_i such that (16) be valid, we assume u as a monogenic function with values in $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$. Then we obtain

$$\mathbf{D}(\mathcal{F}u) = \sum_{i=1}^n (\mathbf{D}A^{(i)} - \mathbf{D}A^{(0)} \cdot \beta_i e_i) \partial_i u - \sum_{i=1}^n \sum_{j=0}^n A^{(j)}(x) \lambda_0 \partial_j(\beta_i) e_i \partial_i u.$$

Therefore the operator \mathcal{F} is associated to operator D if the coefficient of each $\partial_i u$ vanish identically, i.e.,

$$(\mathbf{D}A^{(i)} - \mathbf{D}A^{(0)} \cdot \beta_i e_i) - \sum_{j=0}^n A^{(j)}(x) \lambda_0 \partial_j(\beta_i) e_i = 0 \quad \text{for each } i = 1, \dots, n.$$

Dividing by λ_0 and applying operator \mathbf{D} we have the system

$$\partial_0 A^{(i)} + \alpha_i \beta_i^2 \partial_i A^{(0)} - 2\beta_i \sum_{k=i+1}^n \gamma_{ik} \beta_k \partial_k A^{(0)} = 0 \quad (22)$$

$$\beta_j \partial_j A^{(i)} = 0, \quad \text{for } j \neq i \quad (23)$$

$$\beta_i \partial_i A^{(i)} - \beta_i \partial_0 A^{(0)} - \sum_{j=0}^n A^{(j)} \partial_j(\beta_i) = 0 \quad (24)$$

$$\beta_j \beta_i \partial_j A^{(0)} = 0, \quad \text{for } j \neq i. \quad (25)$$

If $\beta_j, \beta_i \neq 0$, the system (22)-(25) leads to an uncoupled system of equations

$$\beta_i (\partial_i A^{(i)} - \partial_0 A^{(0)}) - \sum_{j=0}^n A^{(j)} \partial_j(\beta_i) = 0. \quad (26)$$

Therefore we have proved the following

Theorem 2. *Consider the operator \mathcal{F} defined by (8) with real valued coefficients $A^{(i)} = A^{(i)}(x_i)$ arbitrarily given and of class \mathbf{C}^1 , then \mathcal{F} is associated to each generalized Cauchy- Riemman operator given by (7) for which the functions $\beta_i = \frac{\lambda_i}{\lambda_0}$, $\lambda_0 \neq 0$ satisfy the system (26).*

Remark 2. *The former theorem generalizes the results appearing in Theorem 3 of [8] in which the author tried the case $n = 3$ in \mathcal{A}_n .*

4 Initial value problems

Using the theory of associated spaces [2, 7, 9] the initial value problem (1), (2) can be solved. In this work, conditions for the associated pair $(\mathcal{F}, \mathbf{D})$ have been obtained in two ways: Given \mathbf{D} , if the equations (18) or (19)-(21) are satisfied, we can get the conditions on the functions $A^{(i)}$ which allow to find \mathcal{F} , reciprocally given \mathcal{F} , through (26), we can determine the operator \mathbf{D} . Thus one obtains immediately that the operator \mathcal{F} sends monogenic functions into monogenic functions.

It is easy to see that solutions of the initial value problem (1), (2) are fixed points of the integro-differential operator (4) and vice versa. In order to apply a fixed point theorem, solutions of $\mathbf{D}u = 0$ must satisfy an interior estimate of first order (see [5, 8, 9]). Therefore the first order derivatives of the solution u , contained in the operator (4) can be estimated. This requirement is achieved if the solutions of $\mathbf{D}u = 0$ are solutions of an elliptical differential equation (see [1, 9]). In our case, such solutions satisfy the equation (10) which is an elliptic equation at least under the condition (11) or (12). Hence we can apply the following theorem (see [7, 9])

Theorem 3. *Suppose \mathcal{F} and \mathbf{D} are a pair of associated operators and the solutions of the associated equation $\mathbf{D}u = 0$ satisfy an interior estimate of first order. Then the initial value problem (1) – (2) is soluble provided the initial solution φ satisfies the condition $\mathbf{D}\varphi = 0$.*

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